Spectral methods for numerical relativity

Philippe Grandclément

Laboratoire Univers et Théorie (LUTH)
CNRS / Observatoire de Paris
F-92195 Meudon, France

philippe.grandclement@obspm.fr

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Numerical relativity

Numerical relativity

Central object the metric : $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu$
Einstein's equations couples the geometry and the energy content. It is a set of 10 highly coupled non-linear equations.

Two families of methods

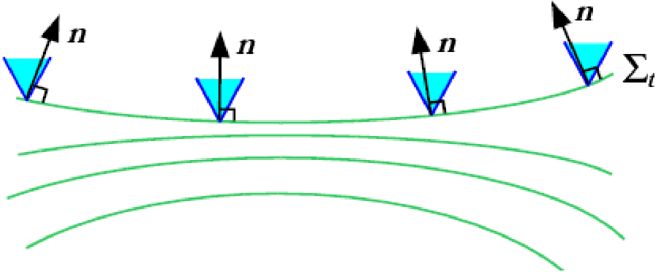
- Analytic : expansion wrt small parameters (v/c, mass ratio etc).
- Use of computers : numerical relativity.

Fields of application

- Coalescence of compact binaries.
- Supernovae explosions.
- Structure of compact objects (magnetized neutrons stars, bosons stars...)
- Critical phenomena.
- Stability of ADS spacetimes (geons)
- and more...

3+1 formalism

Write Einstein equations in a way that is manageable by computers.
It is a way of explicitly splitting time and space.
Spacetime is foliated by a family of spatial hypersurfaces Σ_t.



- Coordinate system of Σ_t : (x_1, x_2, x_3).
- Coordinate system of spacetime : (t, x_1, x_2, x_3).

Metric quantities

The line element reads

$$ds^2 = -\left(N^2 - N^i N_i\right) dt^2 + 2N_i dt dx^i + \gamma_{ij} dx^i dx^j$$

Various functions

- Lapse N, shift $\vec{N}$ and spatial metric γ_{ij}.
- They are all temporal sequences of spatial quantities.
- Lapse and shift are coordinate choice.

Second fundamental form

The extrinsic curvature tensor K_{ij}, which is roughly speaking the time derivative of the metric γ_{ij}.

Projection of Einstein's equations

Type	Einstein	Maxwell
Constraints	Hamiltonian $R + K^2 - K_{ij}K^{ij} = 0$ Momentum : $D_j K^{ij} - D^i K = 0$	$\nabla \cdot \vec{E} = 0$ $\nabla \cdot \vec{B} = 0$
Evolution	$\frac{\partial \gamma_{ij}}{\partial t} - \mathcal{L}_{\vec{B}} \gamma_{ij} = -2NK_{ij}$ $\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\vec{B}} K_{ij} = -D_i D_j N + N\left(R_{ij} - 2K_{ik}K^k_j + KK_{ij}\right)$	$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \mu_0}\left(\vec{\nabla} \times \vec{B}\right)$ $\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$

A two steps problem

Evolution problem

- Given initial value of $\gamma_{ij}\,(t=0)$ and $K_{ij}\,(t=0))$ use the evolution equations to determine the fields at later times.
- Similar to writing Newton's equation as $\partial_t x = v; \partial_t v = f/m$.
- Must ensure stability and accuracy.
- Must choose the lapse and shift in a clever way.

Initial data

- $\gamma_{ij}\,(t=0)$ and $K_{ij}\,(t=0)$ are not arbitrary but subject to the constraint equations.
- Is is a set of four elliptic coupled equations.
- Needs to make the link between a given physical situation and the mathematical objects γ_{ij} and K_{ij}

Both steps are equally important and complicated.

Symmetries in time

Not all numerical relativity has to do with explicit time evolution.

Symmetries

- Stationarity : $\partial_t = 0$.
- Helicoidal Killing vector : $\partial_t = \Omega \partial_\varphi$.
- Periodicity : spectral expansion in time (discrete Fourier transform).

Essentially reduces the system to a elliptic-like one (not always). Many application : (quasi)-circular orbits, structure of compact objects (NS, BS, geons...)...

KADATH

KADATH library

KADATH is a library that implements spectral methods in the context of theoretical physics.

- It is written in C++, making extensive use of object oriented programming.
- Versions are maintained via git.
- Website : *www.kadath.obspm.fr*
- The library is described in the paper : *JCP* **220**, *3334 (2010).*
- Designed to be very modular in terms of geometry and type of equations.
- LateX-like user-interface.
- More general than its predecessor LORENE.

Spectral expansion

Given a set of orthogonal functions Φ_i on an interval Λ, spectral theory gives a recipe to approximate f by

$$f \approx I_N f = \sum_{i=0}^{N} a_i \Phi_i$$

Properties

- the Φ_i are called the basis functions.
- the a_i are the coefficients.
- Multi-dimensional generalization is done by direct product of basis.

Usual basis

- Orthogonal polynomials : Legendre or Chebyshev.
- Trigonometrical polynomials (discrete Fourier transform).

Coefficient and configuration spaces

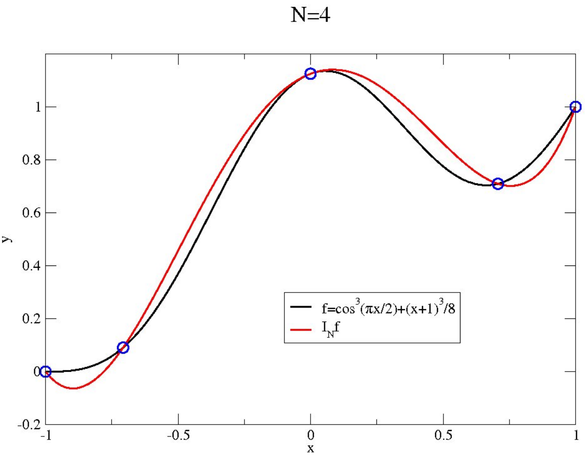
There exist $N+1$ point x_i in Λ such that

$$f(x_i) = I_N f(x_i)$$

Two equivalent descriptions

- Formulas relate the coefficients a_i and the values $f(x_i)$
- Complete duality between the two descriptions.
- One works in the coefficient space when the a_i are used (for instance for the computation of f').
- One works in the configuration space when the $f(x_i)$ are employed (for the computation of $\exp(f)$)

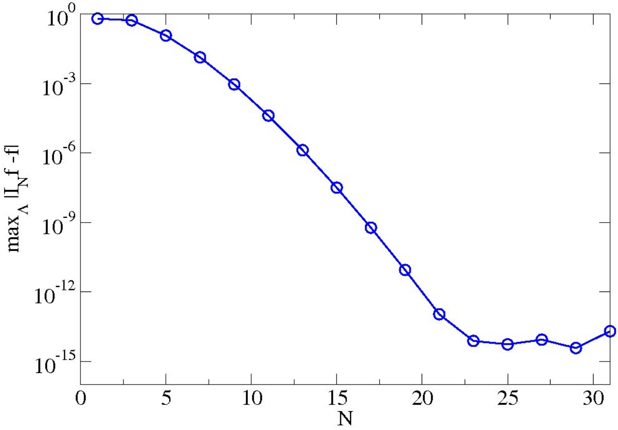
Example of interpolant



Spectral convergence

- If f is $\mathcal{C}^{\infty}$, then $I_N f$ converges to f faster than any power of N.
- Much faster than finite difference schemes.
- For functions less regular (i.e. not $\mathcal{C}^{\infty}$) the error decreases as a power-law.
- Spectral convergence can be recovered using a multi-domain setting.

Convergence



Weighted residual method

Consider a field equation $R = 0$ (ex. $\Delta f - S = 0$). The discretization demands that

$$(R, \xi_i) = 0 \quad \forall i \leq N$$

Properties

- $(,)$ is the same scalar product as the one used for the spectral approximation.
- the ξ_i are called the test functions.
- For the τ-method, the ξ_i are the basis functions.
- Amounts to cancel the coefficients of R.
- Some equations are relaxed and must be replaced by appropriate boundary and matching conditions.

The discrete system

Original system

- Unknowns : tensorial fields.
- Equations : partial derivative equations.

Discretized system

- Unknowns : coefficients $\vec{u}$.
- Equations : algebraic system $\vec{H}\left(\vec{u}\right) = 0$.

Properties

- For a linear system $\vec{H}\left(\vec{u}\right) = 0 \Longleftrightarrow A^i_j u^j = S^i$
- In general $\vec{H}\left(\vec{u}\right)$ is even not known analytically.
- $\vec{u}$ is sought numerically.

Newton-Raphson method

Features

- Starts with an initial guess for $\vec{u}$ and (hopefully) converges to the solution.
- Multi-dimensional generalization of Newton secant method.
- At each iteration : one needs to invert a linear system described by the Jacobian : $Jx = S$.

Automatic differentiation

- Each coefficient becomes a dual number $\langle a, \delta a \rangle$
- Redefine all the arithmetic
  example : $\langle x, \delta x \rangle \times \langle y, \delta y \rangle = \langle x \times y, x \times \delta y + \delta x \times y \rangle$
- One can show that
  $$\vec{H}\left(\langle \vec{u}, \delta\vec{u} \rangle\right) = \left\langle \vec{H}\left(\vec{u}\right), \mathbf{J}\left(\vec{u}\right) \times \delta\vec{u} \right\rangle$$
- The Jacobian is obtained column by column by taking all the possible values of $\delta\vec{u}$.

Numerical resources

Consider N_u unknown fields, in N_d domains, with d dimensions. If the resolution is N in each dimension, the Jacobian is an $m \times m$ matrix with :

$$m \approx N_d \times N_u \times N^d$$

For $N_d = 5$, $N_u = 5$, $N = 20$ and $d = 3$, one reaches $m = 200\,000$

Solution

- The matrix is distributed on several processors.
- Easy because the Jacobian is computed column by column.
- The library `SCALAPACK` is used to invert the distributed matrix.

- $d = 1$ problems : sequential.
- $d = 2$ problems : 100 processors (mesocenters).
- $d = 3$ problems : 1000 processors (national supercomputers).

Linear solver alternatives

It is the most demanding part.
Difficult to go beyond $m = 200\,000$ with SCALAPACK.
Other libraries available ?

Iterative techniques

- Solution sought iteratively.
- Need only to compute products $J \times x$
- But convergence is far from being guaranteed.
- For KADATH , the matrix is dense, lack of preconditioner ...
- Not much success so far...

GEONS

Geons (with G. Martinon)

Original geons

- Idea from Wheeler 1955 : Gravitational Electromagnetic entity.
- Model for elementary particles.
- EM field coupled to GR.
- Not the right model but a fruitful idea.

Gravitational geons

- Packet of GW kept coherent by its own gravitational field.
- Not possible in asymptotically flat spacetime
- A "small" packet disperses.
- A "big" packet collapses to a black hole.
- Different with a cosmological constant...

Anti De Sitter (ADS) spacetimes

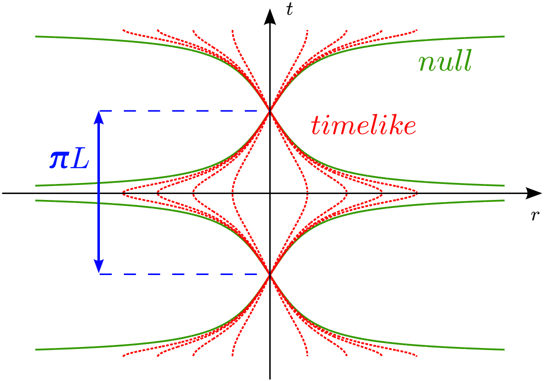
- Maximally symmetric spacetime with a negative cosmological constant.
- Einstein's equations : $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, with $\Lambda < 0$.
- Various coordinate systems, for instance :
    - Static coordinates

$$\mathrm{d}s^2 = -\left(1+\frac{r^2}{L^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1+\frac{r^2}{L^2}\right)} + r^2\mathrm{d}\Omega^2$$

    - Isotropic coordinates

$$\mathrm{d}s^2 = -\left(\frac{1+\frac{r^2}{L^2}}{1-\frac{r^2}{L^2}}\right)^2\mathrm{d}t^2 + \left(\frac{2}{1-\frac{r^2}{L^2}}\right)^2\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right)$$

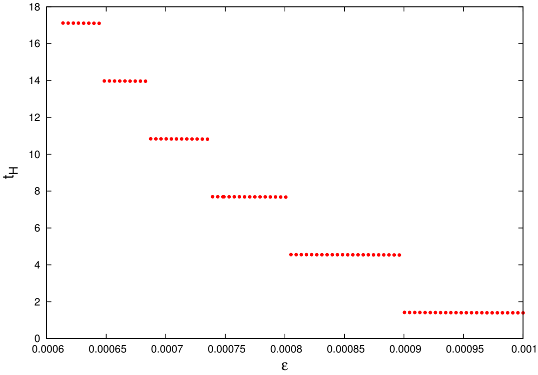
Radial geodesics



$\Lambda < 0$ prevents the fields to be radiated away $\Longrightarrow$ gravitational geons could exist.

Stability of ADS

Due to the attractive effect of $\Lambda < 0$ a small perturbation always collapses to a black hole. ADS is generically unstable.

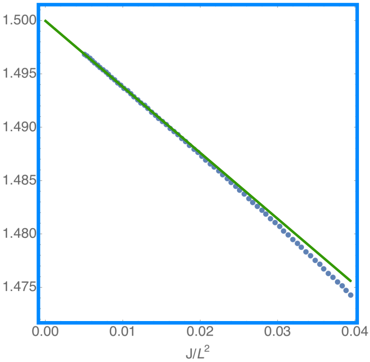


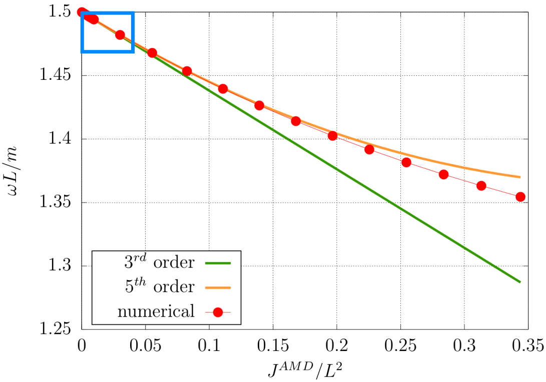
If they exist geons can be seen as island of stability of ADS.

Computing geons with KADATH

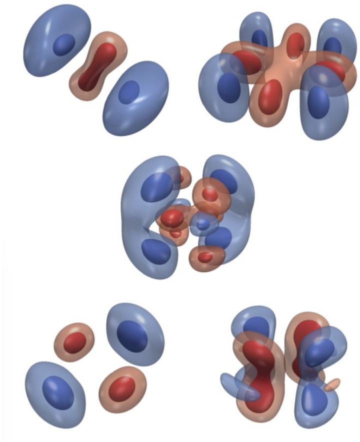
- Extract the background : $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$.
- Geons with helicoidal symmetry : 3D in the corotating frame.
- Regularization of diverging quantities at the boundary of ADS.
- Solve the 3+1 equations using maximal slicing and spatial harmonic gauge.
- The first guess is given by a perturbation theory on maximally symmetric spacetimes (Kodama-Ishibashi-Seto formalism).
- Sequences are constructed by slowly increasing the amplitude.

Geons with $(l, m, n) = (2, 2, 0)$





Other families of helicoidal geons



The cosmological constant as a tool

- By preventing outgoing radiation : quasi-periodic $\Longrightarrow$ exactly periodic.
- Proven in the case of a massive scalar field.
- Possible application : computation of a non-coalescing binary black hole configuration where the GW emission is in equilibrium.
- Study the limit $\Lambda \to 0$ should give information about outgoing gravitational waves.

Conclusions

Kadath

- Efficient tool for (quasi)-stationary spacetimes.
- Many applications : neutron star models, boson stars, binary systems, oscillatons, geons...
- The future is to do "true" time evolutions.

Geons

- First trustworthy numerical computations.
- Axisymmetric and periodic geons are also available
- Future : compute non-coalescing binary black holes with $\Lambda < 0$.