A well-balanced and all regime scheme for stratified compressible flows

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Abstract
Convection in the interior of stars, giant (exo-)planets or in thermohydraulic problems require an hydrodynamical approach using Euler equations with gravity. There are two known issues. First at low Mach regime, when the flow is slow compared to sound waves, Godunov-type solvers lose accuracy and require a lot of iterations. Secondly, near equilibrium problems are difficult to simulate because standard techniques do not preserve steady states. We propose a splitted and well-balanced scheme to overcome these issues that can be qualified of "all regime".

Introduction

We are interested in convective simulations in a stratified medium. To do so we need to solve Euler system in its conservative formulation with the following source terms : gravity $\mathbf{g}$, heat conduction flux $\kappa\nabla_{\mathbf{x}}T$ and viscous flux $\mathbb{S}$.

$$\begin{aligned}
&\partial_t\rho \quad + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}) &&= 0\\
&\partial_t(\rho\mathbf{u}) + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u} + p(\rho,e)\mathbf{I}_3) &&= \rho\mathbf{g} \quad + \nabla_{\mathbf{x}}\cdot\mathbb{S}\\
&\partial_t(\rho E) + \nabla_{\mathbf{x}}\cdot((\rho E + p(\rho,e))\mathbf{u}) &&= \rho\mathbf{g}\cdot\mathbf{u} + \nabla_{\mathbf{x}}\cdot(\mathbb{S}\mathbf{u}) + \nabla_{\mathbf{x}}\cdot(\kappa\nabla_{\mathbf{x}}T)\\
&\qquad \rho E = \rho e + \frac{1}{2}\rho\|\mathbf{u}\|^2
\end{aligned} \tag{1}$$

In our case there are several difficulties about discretization of this system linked to stability and precision issues :

1. Godunov-type colocated schemes have too much numerical viscosity at low Mach number and low spatial resolution [7, 5, 3]
2. the Courant-Friedrichs-Levy stability condition for explicit methods is very restrictive especially when the fluid speed is small compared to sound speed.
3. continuous hydrostatic equilibrium

$$\mathbf{u} = \mathbf{0} \qquad \nabla_{\mathbf{x}}p = \rho\mathbf{g} \qquad \nabla_{\mathbf{x}}\cdot(\kappa\nabla_{\mathbf{x}}T) = 0 \tag{2}$$

has to be captured at discrete level [4]. Standard techniques of discretizing the gravity source term (centered discretization or fractional steps) cannot ensure the balance between gravity and pressure gradient and thus leads to non-physical perturbations.

Main Objectives

Define a Finite Volume scheme for the stratified Euler system with the following properties

- accurate in the low Mach regime
- uniform time step with respect to the Mach number
- well-balanced discretization of the gravity source term
- deal with any pressure law

The two other source terms, heat conduction and viscosity, are discretized at order one with standard finite differences formulas.

Derivation of the scheme

Operator splitting

The scheme [1] is based on an operator splitting which consists in decoupling the acoustic dynamic from the transport one. By developing fluxes terms in Euler equations, the two systems write

- Acoustic+gravity system

$$\begin{aligned}
&\partial_t\rho \quad + \rho\ \nabla_{\mathbf{x}}\cdot\mathbf{u} &&= 0\\
&\partial_t(\rho\mathbf{u}) + \rho\mathbf{u}\,\nabla_{\mathbf{x}}\cdot\mathbf{u} + \nabla_{\mathbf{x}}\cdot(p(\rho,e)I_3) &&= \rho\mathbf{g}\\
&\partial_t(\rho E) + \rho E\nabla_{\mathbf{x}}\cdot\mathbf{u} + \nabla_{\mathbf{x}}\cdot(p(\rho,e)\mathbf{u}) &&= \rho\mathbf{g}\cdot\mathbf{u}
\end{aligned} \tag{3}$$

  - fast waves, eigenvalues in direction $\boldsymbol{\omega}$: $\lambda^0_{\boldsymbol{\omega}} = 0, \quad \lambda^\pm_{\boldsymbol{\omega}} = \pm c$
  - implicit or explicit discretization $(n \longrightarrow n+1-)$
  - Suliciu-like relaxation on pressure
  - well-balanced discretization of gravity [2]

- Transport system

$$\begin{aligned}
&\partial_t\rho \quad + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\,\rho \quad &&= 0\\
&\partial_t(\rho\mathbf{u}) + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\,\rho\mathbf{u} &&= \mathbf{0}\\
&\partial_t(\rho E) + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\,\rho E &&= 0
\end{aligned} \tag{4}$$

  - slow waves, eigenvalues in direction $\boldsymbol{\omega}$: $\lambda^{0,\pm}_{\boldsymbol{\omega}} = \mathbf{u}\cdot\boldsymbol{\omega}$
  - explicit discretization $(n+1- \longrightarrow n+1)$
  - upwind scheme

Whole step

The combination of the two steps gives the following conservative scheme (omitting diffusion terms)

$$\begin{aligned}
\rho_j^{n+1} &= \rho_j^n \quad - \Delta t\sum_{k\in N(j)}\frac{|\Gamma_{jk}|}{|\Omega_j|}\rho_{jk}^{n+1-}u^*_{jk}\\
(\rho\mathbf{u})_j^{n+1} &= (\rho\mathbf{u})_j^n - \Delta t\sum_{k\in N(j)}\frac{|\Gamma_{jk}|}{|\Omega_j|}\left((\rho\mathbf{u})_{jk}^{n+1-}u^*_{jk} + \Pi^{*,\theta}_{jk}\mathbf{n}_{jk}\right) + \frac{\Delta t}{2}\sum_{k\in N(j)}\frac{1}{2}\left(\rho_j^n + \rho_k^n\right)g^n_{jk}\mathbf{n}_{jk}\\
(\rho E)_j^{n+1} &= (\rho E)_j^n - \Delta t\sum_{k\in N(j)}\frac{|\Gamma_{jk}|}{|\Omega_j|}\left((\rho E)_{jk}^{n+1-} + \Pi^{*,\theta}_{jk}\right)u^*_{jk} \quad + \frac{\Delta t}{2}\sum_{k\in N(j)}\frac{1}{2}\left(\rho_j^n + \rho_k^n\right)g^n_{jk}u^*_{jk}
\end{aligned}$$

$$u^*_{jk} = \frac{1}{2}\left(\mathbf{u}^\flat_j + \mathbf{u}^\flat_k\right)\cdot\mathbf{n}_{jk} - \frac{1}{2a_{jk}}\left(\Pi^\flat_k - \Pi^\flat_j - \frac{|\Omega_j|}{|\Gamma_{jk}|}\frac{1}{2}\left(\rho_j + \rho_k\right)g^n_{jk}\right)$$

$$\Pi^{*,\theta}_{jk} = \frac{1}{2}\left(\Pi^\flat_j + \Pi^\flat_k\right) - \frac{a_{jk}\theta_{jk}}{2}\left(\mathbf{u}^\flat_k - \mathbf{u}^\flat_j\right)\cdot\mathbf{n}_{jk}$$

Here the exponent $\flat$ is either equal to n for an explicit treatment of the acoustic step or $n+1-$ for an implicit step.

- Discrete steady states are defined by 5 and are preserved up to the machine precision,

$$\frac{|\Gamma_{jk}|}{|\Omega_j|}\left(\Pi_k - \Pi_j\right) = \frac{1}{2}\left(\rho_k + \rho_j\right)g_{jk} \tag{5}$$

- The factor θ_{jk} in the pressure flux $\Pi^{*,\theta}_{jk}$ is called the low Mach correction. It comes from an analysis of the truncation error. It aims to decrease the numerical viscosity in the low Mach regime hence to increase the accuracy of the scheme.

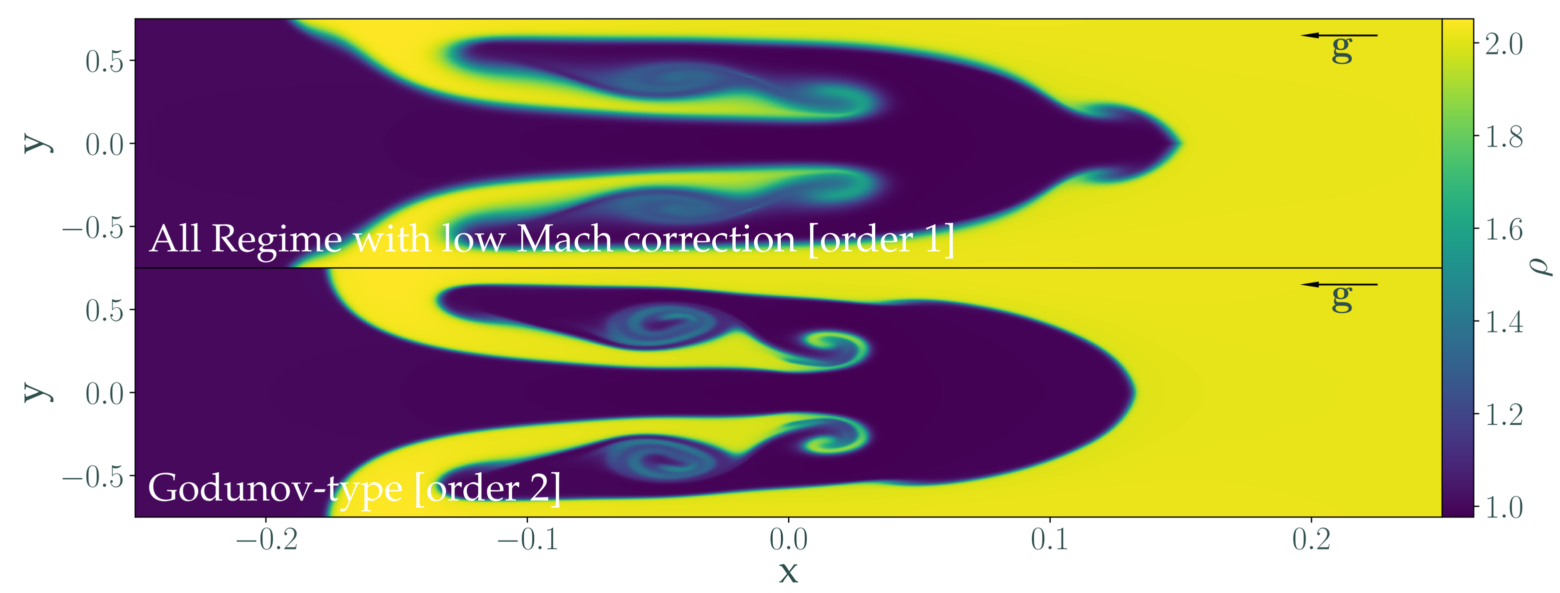
Numerical results

Isothermal atmosphere test case [equilibrium test case]

This a one dimensional test case (ny=500) that consists in simulating a column of isothermal atmosphere at rest. Thus the term $\nabla_{\mathbf{x}}\cdot\{\kappa\nabla_{\mathbf{x}}T\}$ trivially vanishes. We know that density and pressure profiles are exponential and hence subject to spurious movement. Results show an error on the vertical momentum around 10^{-14} as expected.

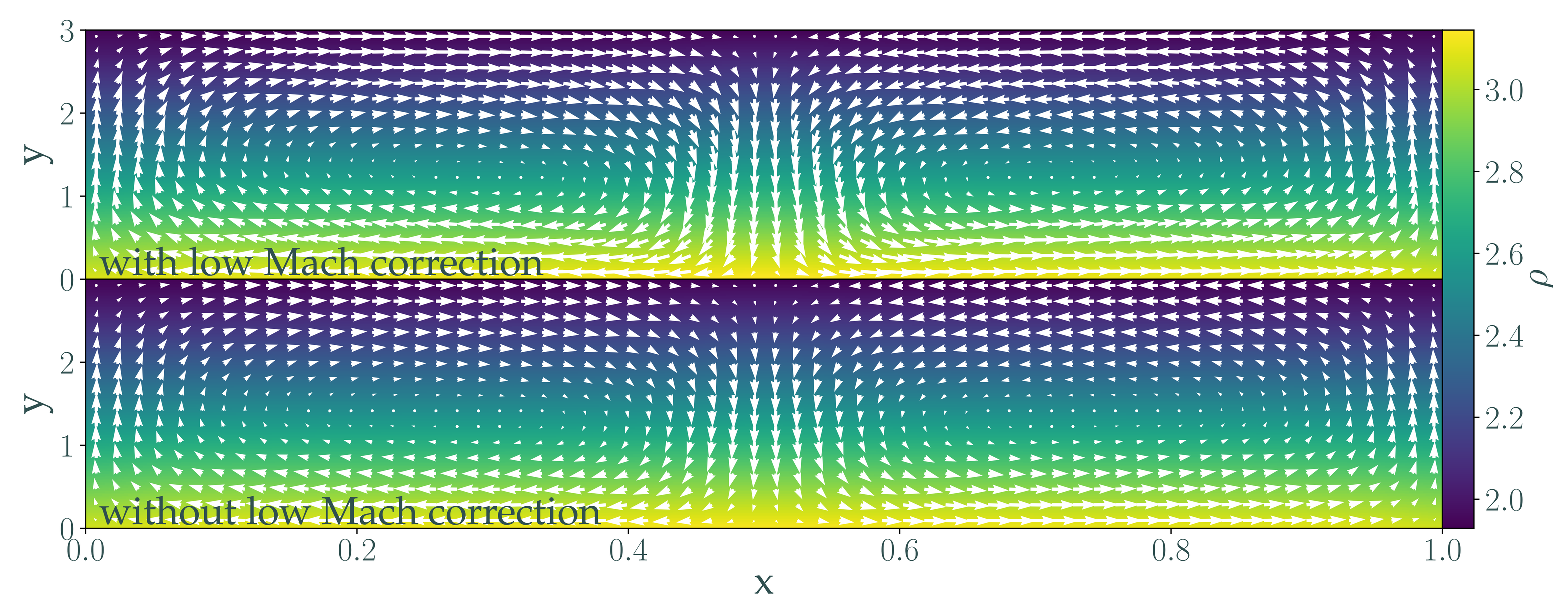
Rayleigh-Taylor test case [out of equilibrium test case]

This a two dimensional test case (nx=300, ny=900) where a dense gas is placed above a lighter one. It results an instable equilibrium that is perturbed in velocity. Diffusion terms are disabled. The peak observed with the low Mach correction can also be seen with a third order Godunov-type scheme (see JOANNA code [8]).



Convection — Rayleigh-Bénard test case [near equilibrium test case]

This is a two dimensional test case [6] (nx=640, ny=160) that consists in simulating a convection cell. The background atmosphere is at rest with a linear temperature profile. It is perturbed in the velocity field to develop convection cells. One can observe less diffusion in the velocity field thanks to the low Mach correction.



Conclusion

We presented a Finite Volume scheme with gravity source term able to preserve steady states of the form 5. In addition of the well-balancing property, the low Mach correction allows to study accuratly flows near hydrostatic equilibrium. The current scheme is only order one in time and space. Future work will consist in increasing the order of the scheme by using for instance Discontinuous Galerkin methods.

References

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